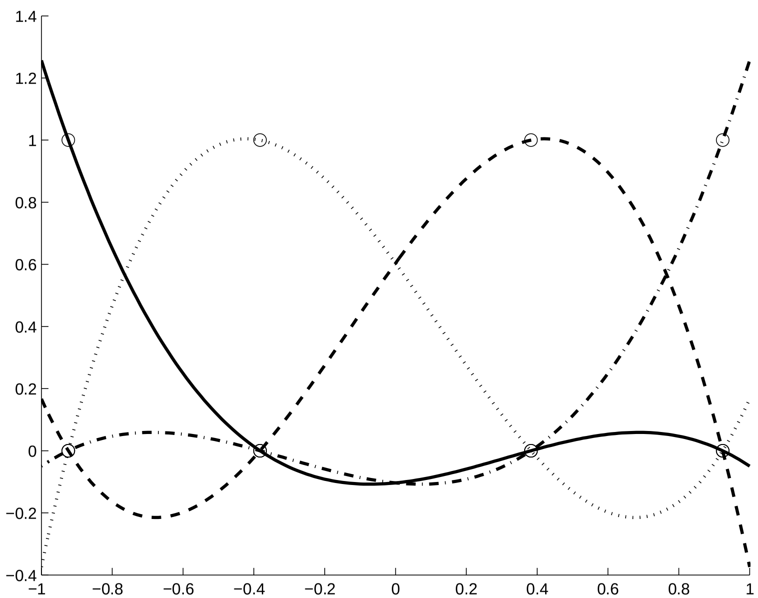
Desanka Radunović – NUMERIČKE METODE

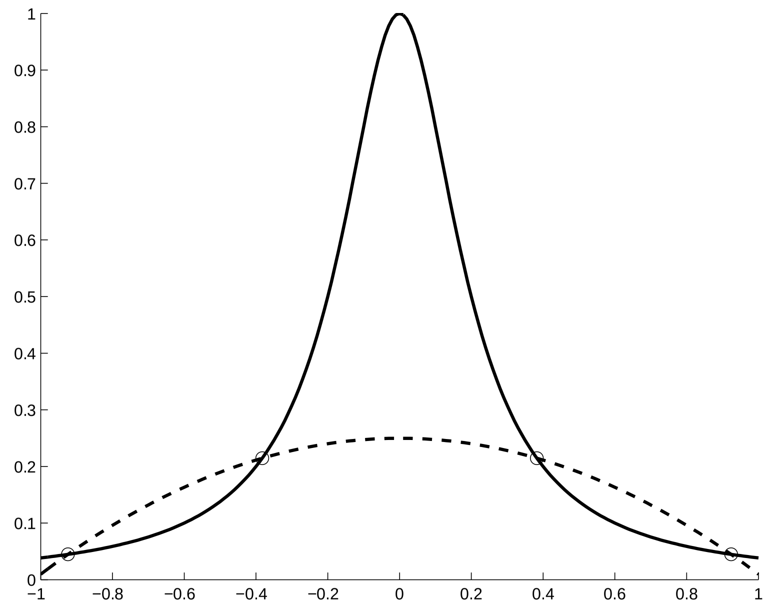
Interpolacija i integracija

Lagrange-ov interpolacioni polinom

$$L_n(x) = \sum_{i=0}^{n} c_i x^i, \qquad L_n(x_k) = f(x_k), \qquad k = 0, \ldots, n.$$

$$L_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i) = \sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \right) f(x_i)$$





$$L_n(x) = \sum_{i=0}^{n} \frac{\omega_{n+1}(x) f(x_i)}{(x - x_i)\omega'_{n+1}(x_i)}, \qquad \omega_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$$

greška

$$f(\overline{x}) - L_n(\overline{x}) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(\overline{x})$$

$$L_{i,\ldots i+k}(x) = \frac{1}{x_{i+k} - x_i}\Big((x - x_i)L_{i+1,\ldots i+k}(x) - (x - x_{i+k})L_{i,\ldots i+k-1}(x)\Big)$$

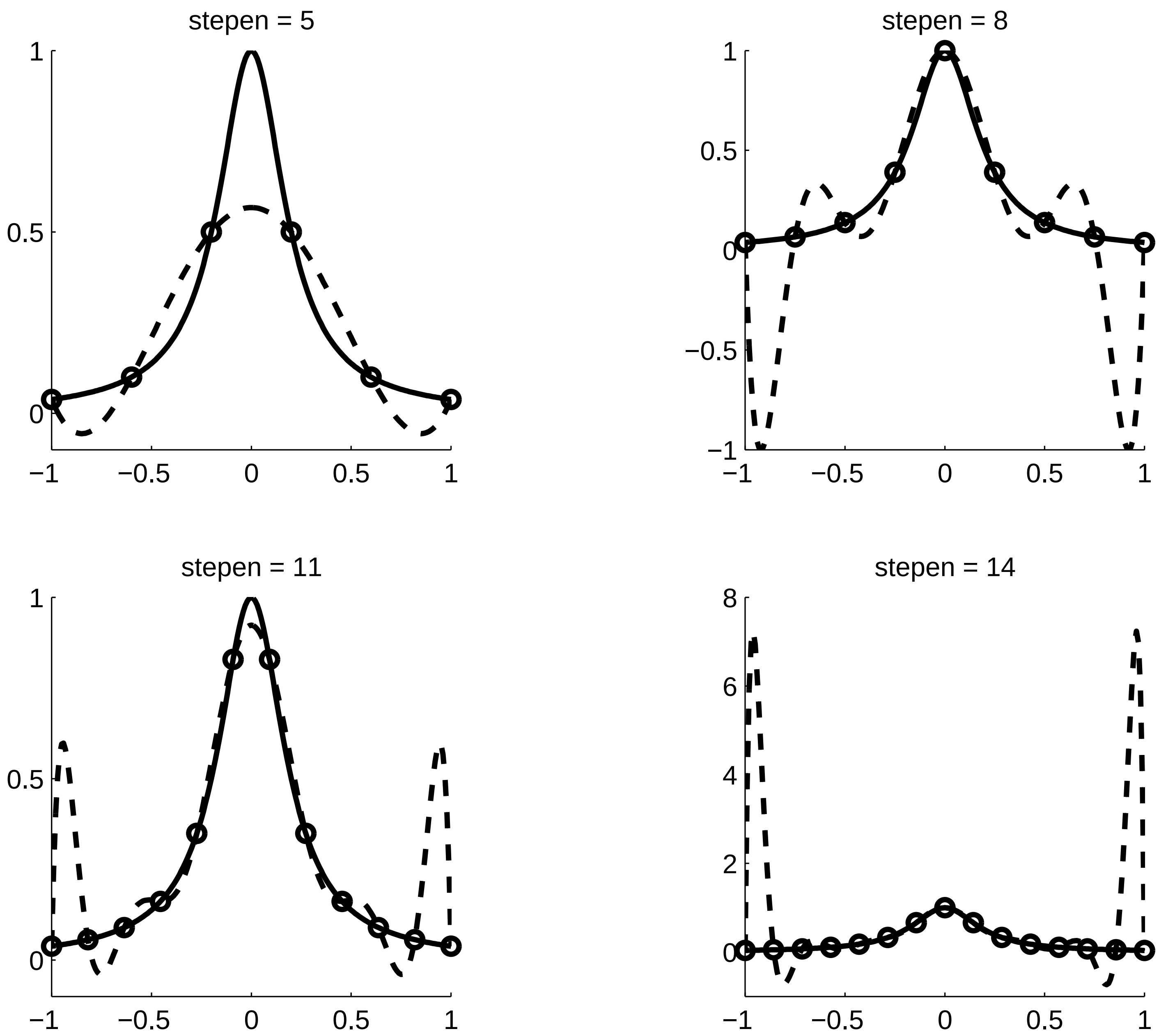
Neville-ov algoritam

$$C_{i0} = D_{i0} = f_i, \quad i = 0, \ldots n,$$

$$1 \le k \le i, \qquad C_{ik} = \tfrac{\overline{x} - x_i}{x_i - x_{i-k}}\big(D_{i,k-1} - C_{i-1,k-1}\big),$$

$$D_{ik} = \tfrac{\overline{x} - x_{i-k}}{x_i - x_{i-k}}\big(D_{i,k-1} - C_{i-1,k-1}\big),$$

$$C_{ik} = T_{ik} - T_{i,k-1}, \qquad D_{ik} = T_{ik} - T_{i-1,k-1}, \qquad T_{ik} = \sum_{j=0}^{k} C_{ij}$$



interpolacija funkcije

$$\frac{1}{1+25x^2}$$

polinomima različitog stepena

Newton-ov polinom sa podeljenim razlikama

$$L_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \ldots, x_n](x - x_0) \cdots (x - x_{n-1})$$

greška $$f(\overline{x}) - L_n(\overline{x}) = \omega_{k+1}(x) f[\overline{x}, x_0, \ldots, x_k]$$

podeljene razlike

$$f[x_{i_0}] = f(x_{i_0}), \qquad f[x_{i_0}, \ldots, x_{i_k}] = \frac{f[x_{i_1}, \ldots, x_{i_k}] - f[x_{i_0}, \ldots, x_{i_{k-1}}]}{x_{i_k} - x_{i_0}}$$

$$f[x_0, \ldots, x_k] = \sum_{i=0}^{k} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j \neq i}}^{k} (x_i - x_j)}, \qquad f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

konačne razlike: za $x_{i+1} - x_i = h$ $f[x_i, \ldots, x_{i+k}] \longrightarrow \Delta^k f_i$

$$\Delta f_i = f_{i+1} - f_i, \qquad \Delta^k f_i = \Delta(\Delta^{k-1} f_i) = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i,$$

$$\Delta^k f_i \quad = \quad \nabla^k f_{i+k} \quad = \quad \delta^k f_{i+\frac{k}{2}}$$

razlika unapred	razlika unazad	centralna razlika

$$\Delta^k f_i = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f_{i+k-j}, \qquad \Delta^k f_i = k!\, h^k f[x_0, \ldots, x_k] = h^k f^{(k)}(\xi)$$

Newton-ovi polinomi sa konačnim razlikama

unapred

$$L_n(x_0+qh) = f_0+q\Delta f_0+\frac{q(q-1)}{2!}\Delta^2 f_0+\cdots+\frac{q(q-1)\cdots(q-n+1)}{n!}\Delta^n f_0,$$

greška

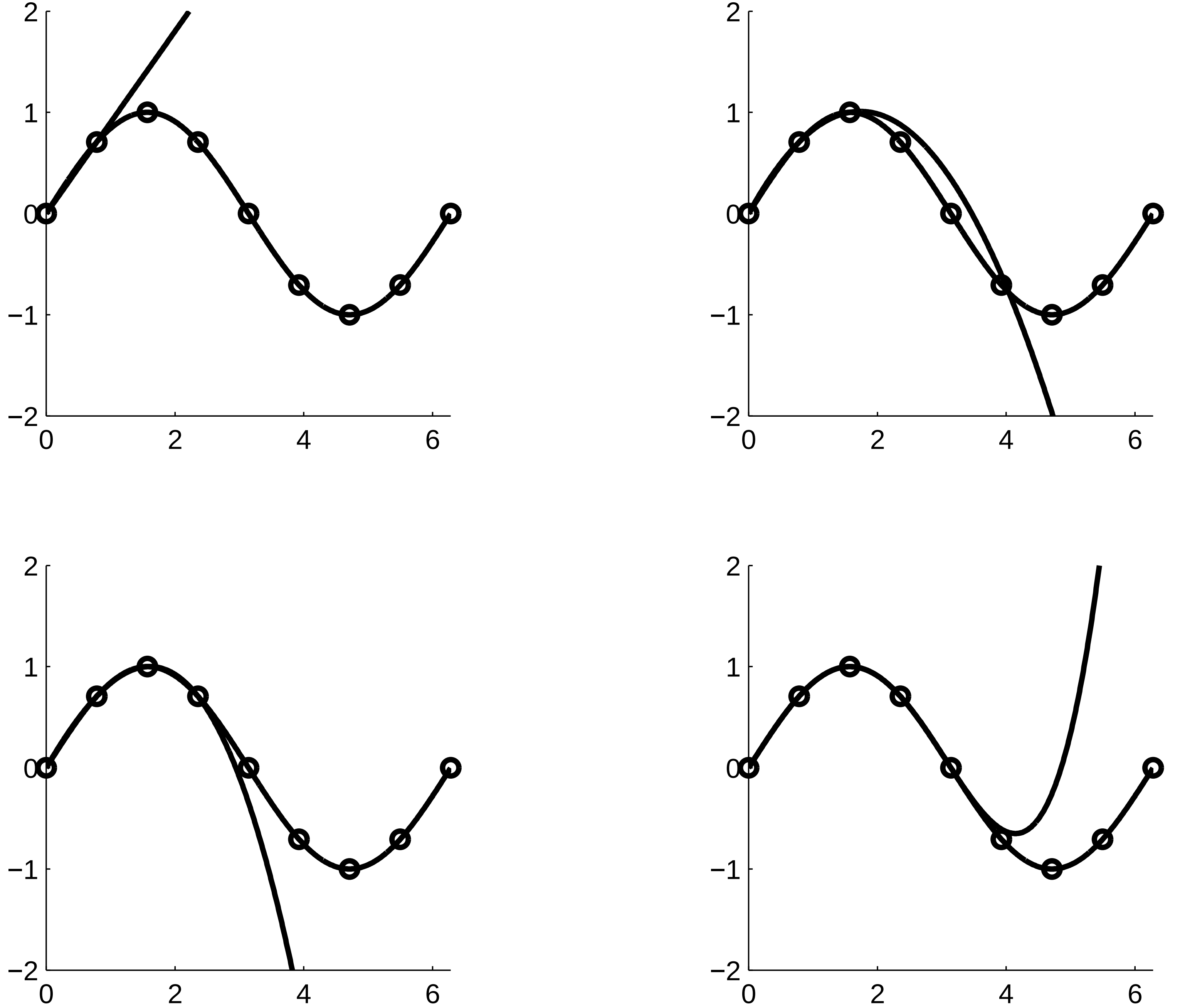
$$f(x_0+qh)-L_n(x_0+qh) = \frac{q(q-1)\cdots(q-n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi), \qquad x_0 \le \xi \le x_n$$

unazad

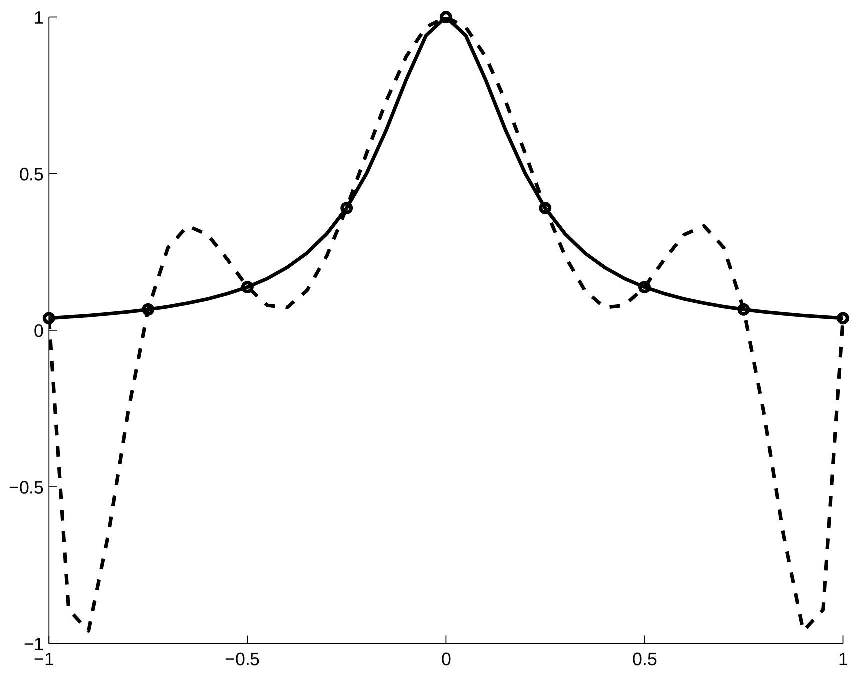
$$L_n(x_0+qh) = f_0+q\Delta f_{-1}+\frac{q(q+1)}{2!}\Delta^2 f_{-2}+\cdots+\frac{q(q+1)\cdots(q+n-1)}{n!}\Delta^n f_{-n}$$

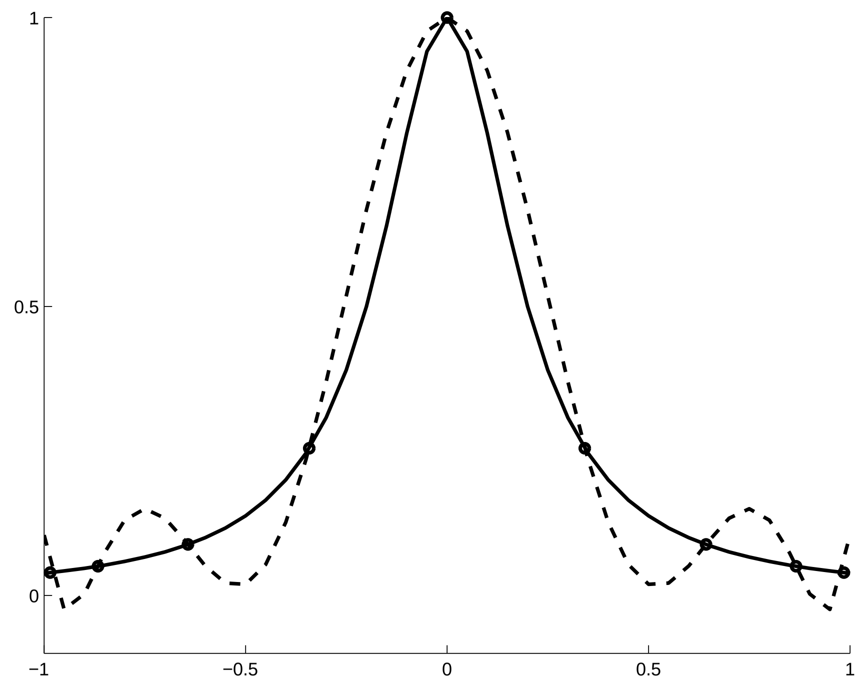
greška

$$f(x_0+qh)-L_n(x_0+qh) = \frac{q(q+1)\cdots(q+n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi), \qquad x_{-n} \le \xi \le x_0$$


2
1
0
−1
−2
0
2
4
6

Optimalan izbor čvorova interpolacije





ravnomerni raspored

nule Čebiševljevog polinoma

$$k = 0, \ldots, n, \qquad x_k = \frac{b-a}{n}$$

$$x_k = \frac{b+a}{2} + \frac{b-a}{2} \cos \frac{2k+1}{2(n+1)}$$

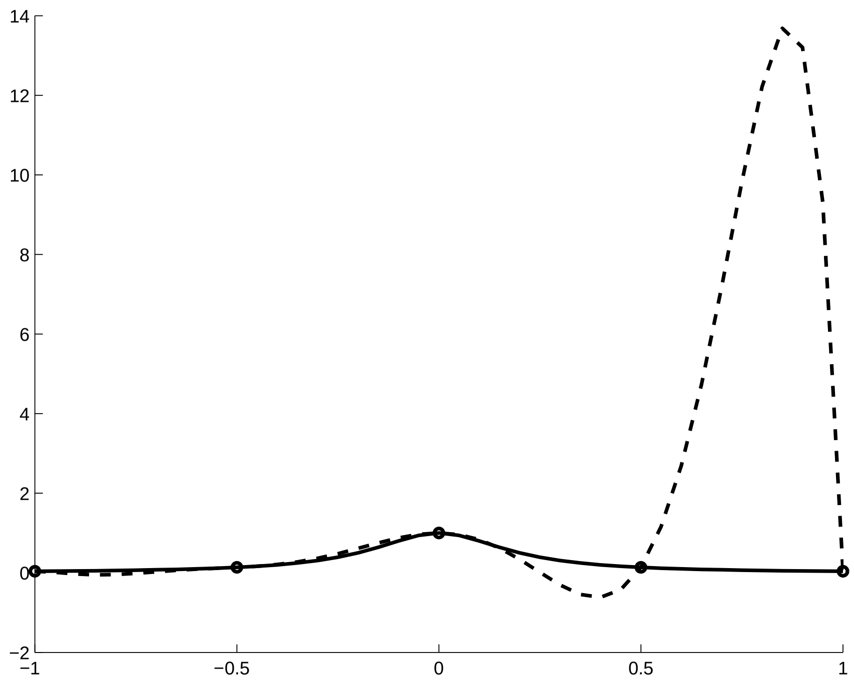
Hermite-ov interpolacioni polinom

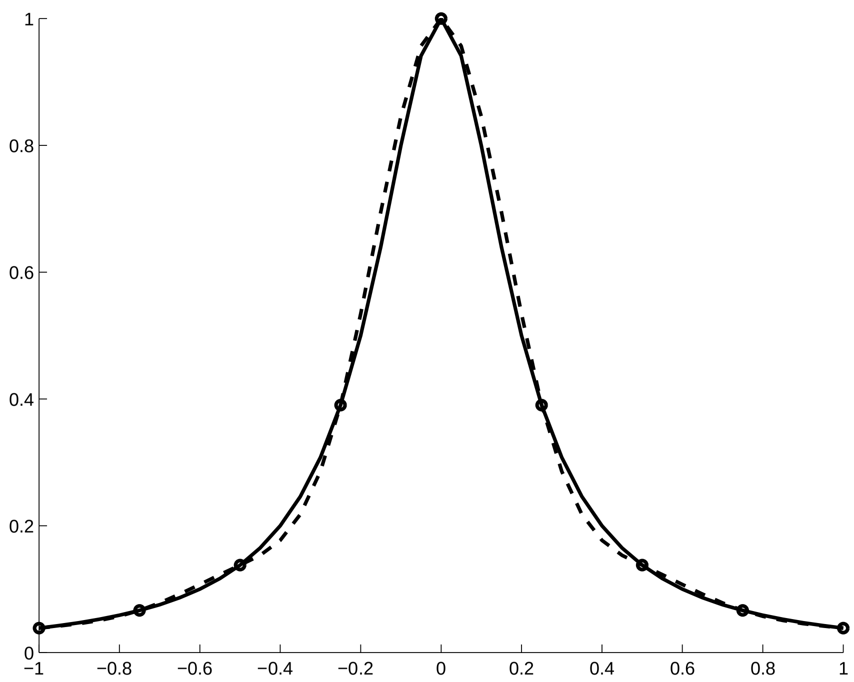
$$\boxed{P_n^{(k)}(x_i) = f_i^k, \qquad k = 0, \ldots, n_i - 1, \quad i = 0, \ldots, m, \qquad n = \sum_{i=0}^{m} n_i - 1}$$

$$\begin{aligned}
P_n(x) &= f(x_0) + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_0](x - x_0)^2 + \cdots \\
&+ f[\underbrace{x_0, \ldots, x_0}_{n_0 \text{ puta}}](x - x_0)^{n_0 - 1} + f[\underbrace{x_0, \ldots, x_0}_{n_0 \text{ puta}}, x_1](x - x_0)^{n_0} \\
&+ f[\underbrace{x_0, \ldots, x_0}_{n_0 \text{ puta}}, x_1, x_1](x - x_0)^{n_0}(x - x_1) + \cdots \\
&+ f[\underbrace{x_0, \ldots, x_0}_{n_0 \text{ puta}}, \ldots, \underbrace{x_m, \ldots, x_m}_{n_m \text{ puta}}](x - x_0)^{n_0} \cdots (x - x_m)^{n_m - 1}
\end{aligned}$$

$$f[\underbrace{x_i, \ldots, x_i}_{(p+1) \text{ puta}}] = \frac{f^{(p)}(x_i)}{p!}$$

Interpolacija funkcije $\frac{1}{1+25x^2}$





Hermite-ovim polinomom (2, 3, 2, 1, 1), kubnim splajnom

Splajn interpolacija

- $S_\triangle^m \in \mathcal{C}^{m-1}[a,b]$
- $S_\triangle^m$ je polinom stepena m na intervalima podele $\triangle = \{a = x_0 < \cdots < x_n = b\}$

$$S_\triangle^3(x) \equiv S_\triangle(x), \qquad S_\triangle(f;x_i) = f(x_i), \quad i = 0,\ldots,n,$$

(*i*) $S''_\triangle(f;a) = S''_\triangle(f;b) = 0$

(*ii*) $f(x)$ i $S_\triangle(f;x)$ su periodične funkcije na $[a,b]$

(*iii*) $f'(a) = S'_\triangle(f;a), \quad f'(b) = S'_\triangle(f;b)$

svojstvo minimalnosti $\qquad \|f\| \geq \|S_\triangle\| \quad$ za $\quad \|f\|^2 = \int_a^b \big(f''(x)\big)^2\,dx$

$$\mu_i M_{i-1} + 2M_i + \nu_i M_{i+1} = \lambda_i, \qquad i = 1, \ldots, n-1, (n)$$

$$\nu_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \qquad \mu_i = \frac{h_i}{h_i + h_{i+1}} = 1 - \nu_i, \qquad \lambda_i = 6\, f[x_{i-1}, x_i, x_{i+1}]$$

(i)

$$\mu_0 = \nu_0 = \lambda_0 = 0, \qquad \mu_n = \nu_n = \lambda_n = 0,$$

(ii)

$$\nu_n = \frac{h_1}{h_n + h_1}, \qquad \mu_n = 1 - \nu_n, \qquad \lambda_n = 6\, f[x_{n-1}, x_n, x_1]$$

(iii)

$$\mu_0 = 0, \qquad \nu_0 = 1, \qquad \lambda_0 = \frac{6}{h_1}\big(f[x_0, x_1] - f_0'\big),$$

$$\mu_n = 1, \qquad \nu_n = 0, \qquad \lambda_n = \frac{6}{h_n}\big(f_n' - f[x_{n-1}, x_n]\big).$$

Dvodimenziona interpolacija

$$L_n(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} (x-x_0)\cdots(x-x_{i-1})(y-y_0)\cdots$$
$$(y-y_{j-1})\, f[x_0,\ldots,x_i; y_0,\ldots,y_j]$$

$$x_i = x_0 + i\,h, \qquad y_j = y_0 + j\,k, \qquad p = \frac{x-x_0}{h}, \quad q = \frac{y-y_0}{k}$$

$$f[x_0,x_1;y_0] = \frac{1}{h}\Delta^{1+0} f(x_0,y_0), \qquad f[x_0,x_1,x_2;y_0] = \frac{1}{2!h^2}\Delta^{2+0} f(x_0,y_0),$$
$$f[x_0,x_1;y_0,y_1] = \frac{1}{hk}\Delta^{1+1} f(x_0,y_0), \qquad \cdots$$

$$L_n(x_0+ph, y_0+qk) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{i+j=k} \binom{k}{i} p\,(p-1)\ldots(p-i+1)$$
$$q\,(q-1)\ldots(q-j+1)\,\Delta^{i+j} f(x_0,y_0)$$

Numeričko diferenciranje i integracija

$$\boxed{f^{(k)}(x) \approx L_n^{(k)}(x)}$$

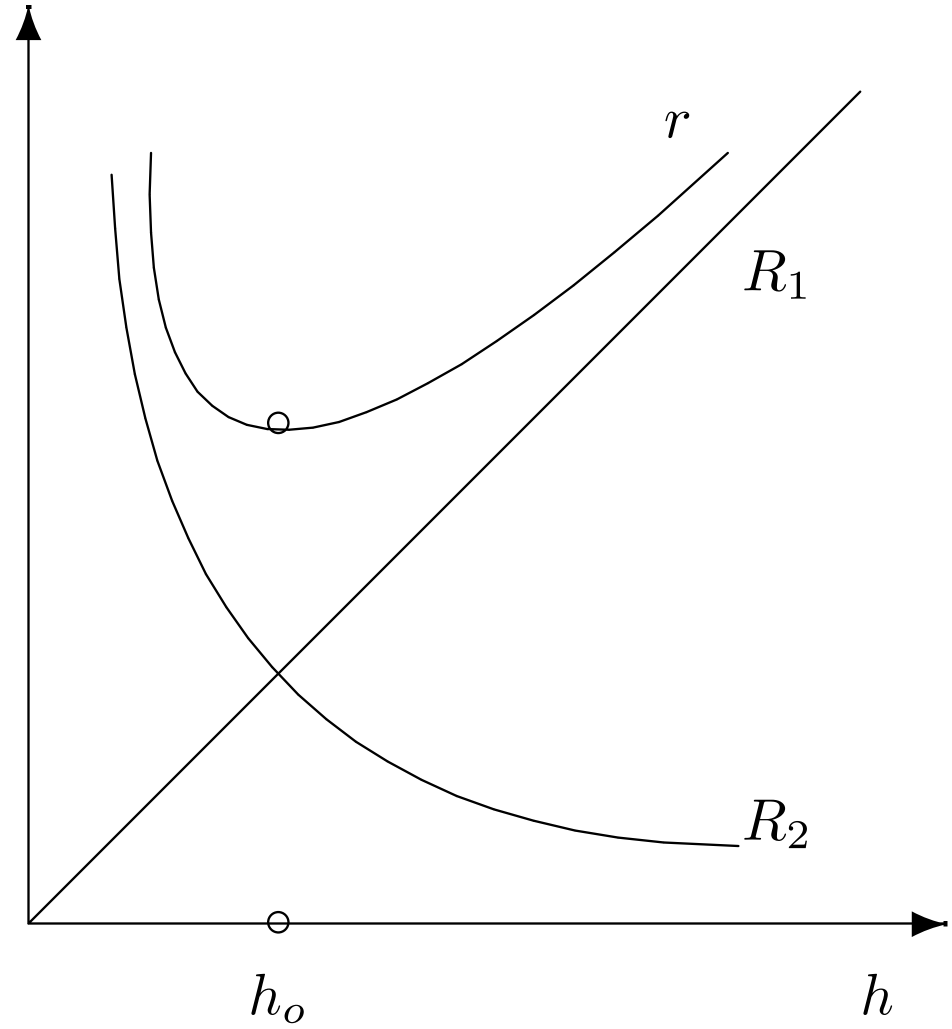
$$f'(x_0) \approx \frac{1}{h} \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \Delta^j f_0, \qquad f'(x_0) \approx \frac{1}{h} \sum_{j=1}^{n} \frac{\Delta^j f_{-j}}{j}, \qquad \big(= L_n'(x_0) \big)$$

greška

$$f^{(k)}(x) - L_n^{(k)}(x) = \big(f(x) - L_n(x)\big)^{(k)} = \big(f[x, x_0, \ldots, x_n]\omega_{n+1}(x)\big)^{(k)}$$

$$= \sum_{j=0}^{k} \frac{k!}{(k-j)!} \frac{f^{(n+j+1)}(\xi)}{(n+j+1)!} \omega_{n+1}^{(k-j)}(x)$$

Zavisnost greške numeričkog diferenciranja od koraka



$$f'(x_0) \approx \frac{f(x_1) - f(x_0)}{h}$$

$$|R| = |R_1 + R_2| \leq \frac{M_2 h}{2} + \frac{2E}{h} \equiv r(h)$$

$$h_o = 2\sqrt{\frac{E}{M_2}}$$

$$|R| \leq r(h_o) = 2\sqrt{M_2 E}$$

Newton-Cotes-ove kvadraturne formule

$$\boxed{I(f) = \int_a^b p(x) f(x)\,dx = \int_a^b p(x)\big(L_n(x) + r_n(x)\big)\,dx}$$

$$I(f) \approx S_n(f) = \int_a^b p(x)\,L_n(x)\,dx = \frac{b-a}{2}\sum_{i=0}^{n} c_i f(x_i)$$

$$c_i = \int_{-1}^{1} \bar{p}(t)\Big(\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{t-t_j}{t_i-t_j}\Big)\,dt, \qquad x = \frac{b+a}{2} + \frac{b-a}{2}\,t$$

$$R_n(f) = \int_a^b p(x)\,r_n(x)\,dx = \int_a^b p(x)\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\omega_{n+1}(x)\,dx, \qquad \xi \in [a,b]$$

Osnovna formula:

pravougaonika

$$S_0(f) = (b-a) f(\tfrac{a+b}{2}), \qquad |R_0(f)| \le \frac{(b-a)^3}{24} \max_{x \in [a,b]} |f''(x)|$$

trapeza

$$S_1(f) = \frac{b-a}{2}\big(f(a) + f(b)\big), \qquad |R_1(f)| \le \frac{(b-a)^3}{12} \max_{x \in [a,b]} |f''(x)|$$

Simpsona

$$S_2(f) = \frac{1}{3}\,\frac{b-a}{2}\,\big(f(a) + 4f(\tfrac{a+b}{2}) + f(b)\big),$$

$$|R_2(f)| \le \frac{1}{90}\left(\frac{b-a}{2}\right)^3 \max_{x \in [a,b]} |f^{(4)}(x)|$$

Opšta formula:

pravougaonika

$$S_0^h(f) = h \sum_{i=1}^{m} f_{i-\frac{1}{2}}, \qquad |R_0^h(f)| \leq \frac{(b-a)h^2}{24} \max_{[a,b]} |f''(\xi)|$$

trapeza

$$S_1^h(f) = \frac{h}{2}(f_0 + 2 \sum_{i=1}^{m-1} f_i + f_m), \qquad |R_1(f)| \leq \frac{(b-a)h^2}{12} \max_{[a,b]} |f''(\xi)|$$

Simpsona

$$S_2^h(f) = \frac{h}{3}(f_0 + 4 \sum_{i=1}^{m} f_{2i-1} + 2 \sum_{i=1}^{m-1} f_{2i} + f_{2m})$$

$$|R_2(f)| \leq \frac{(b-a)h^4}{180} \max_{[a,b]} |f^{(4)}(\xi)|$$

Formule Gauss-ovog tipa

$$\boxed{S_n(f) = \tfrac{b-a}{2} \sum_{i=1}^{n} c_i f(x_i)}$$

$$|R_{2n-1}(f)| \le \frac{1}{(2n)!} \max_{[a,b]} |f^{(2n)}(\xi)| \int_a^b p(x)\omega_n^2(x)\, dx$$

$$\omega_n(x) = \prod_{i=1}^{n} (x - x_i) \qquad \omega_n(x) \equiv Q_n(x)$$

$$(Q_i\,,\,Q_j) = \int_a^b p(x)\, Q_i(x)\, Q_j(x)\, dx = 0, \qquad i \ne j.$$

$$Q_1(x) \stackrel{\text{def}}{=} 0, \qquad Q_0(x) \equiv 1$$

$$Q_{k+1} = \left(x - \frac{(xQ_k\,,\,Q_k)}{(Q_k\,,\,Q_k)} \right) Q_k(x) - \frac{(Q_k\,,\,Q_k)}{(Q_{k-1}\,,\,Q_{k-1})} Q_{k-1}(x), \quad k = 0, 1, \ldots,$$

$$\frac{b-a}{2}\sum_{i=1}^{n} c_i x_i^k = \int_a^b p(x)x^k\,dx, \qquad k = 0,\dots,n-1$$

polinom Legendrea

$$w_n(x) \equiv L_n(x) = \frac{1}{2^n\,n!}\,\frac{d^n}{dx^n}\big((x^2-1)^n\big), \qquad x \in [-1,1], \quad p(x) \equiv 1$$

formula Gaussa

$$\int_{-1}^{1} f(x)\,dx \approx \frac{1}{9}\Big(5f(-\sqrt{\tfrac{3}{5}}) + 8f(0) + 5f(\sqrt{\tfrac{3}{5}})\Big)$$

polinom Čebiševa

$$w_n(x) \equiv T_n(x) = \cos\left(n \operatorname{arccos} x\right), \qquad x \in [-1,1], \quad p(x) \equiv \frac{1}{\sqrt{1-x^2}}$$

formula Čebiševa

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \approx \frac{\pi}{n}\sum_{k=1}^{n} f\big(\cos\tfrac{2k-1}{2n}\pi\big)$$